Module 3, 3303 Fall'14

Functions and Transcendental numbers – Video 1

Popper 03

Review of functions

Definition and examples

Group of invertible functions and composition

Algebraic and Transcendental Numbers – Video 2

Popper 03, continued

Polynomials and the partition created by them

Continued fractions and the search for patterns – Video 3

Popper 03, continued

Continued fractions – what they are are how they can

help us see patterns in some unusual numbers.

Homework 3

Functions are basic math objects. We study them because, among other reasons, they have one very nice feature: if you know your input, your output is fixed, decided for you.

Let me illustrate this with a numerical function:

$f(x) = -3x + 7$

If you chose an *x*, an input, you'll be handed a *y*, a single fixed output, from the calculation…with no choices about which *y* you get. It is the definite output that makes functions easy to use. If I chose $x = 1$, I get $y = 4$ and no other number. If I want 10 as an output, I'd better choose -1 as my input because that's the ONLY way to get a 10 out.

An equation that is not a function, for example:

$$x^2 + y^2 = 4$$

doesn't have this feature.

A **definition** of a function is:

Let *A* and *B* be two sets. A function *f* from *A* to *B* is a rule or relation between A and B that assigns each element *a* in *A* to a unique element *b* in *B*.

So what is a relation? This is short for relationship between two sets. It means there is a discernible connection between elements of set *A* and *B*. It can be as simple as a description in words, a formula, an equation, a table, or a list. Functions are relations with a special extra property as noted above. So a function is not necessarily an equation nor does it necessarily have a formula!

Let's explore some sets and the functions associated with them.

Exploration 1

Let *A* be the set of all social security numbers that have been issued to US citizens between the ages of 15 and 65, and let *B* be the set of all citizens in the US between those ages. The rule that assigns a number to a citizen is a function; you invoke the rule when you apply for the SSN. Note that every social security number in set A is used (see the definition: "to each element") and only one person can use that number legally. If you know the number, you can identify the person.

What if I changed set *B* to everyone who lives in the USA? Well then, with identity theft, this is no longer a function. A number that has been high jacked may be used by TWO people or 3 people! Then, if you know the number, you might not always be able to identify the person who is using it! This causes all KINDS of problems as you can imagine!

Let's make a partial pair of set diagrams to explore this

Yes:

SSN	Legal account holder

No:

SSN	Account holder or Identity Thief

Notice that the definition does NOT say *A* and *B* are sets of numbers! Nor does it say that all relations are functions…only the ones with the given property are functions.

Exploration 2

Another example is assigning kids to their Mom. Suppose Mary has 3 kids under the age of 7. A rule that assigns each child to Mary is a function. Switching the sets and assigning Mary to the kids, however is NOT a function because you'll need to make an additional choice of WHICH kid is the output kid you want.

Let's look at sketches: (aren't these like tree diagrams in a way?)

Exploration 3

Now let's peek in on a harried Mom who has 1 husband, 2 children, and 1 dog to organize. She decided to color code everything – towels, tee shirts, toothbrushes, and food dishes. She picks white for herself and assigns the following colors: Husband Tom: Cougar Red (Class of 1996), son Mike: blue, daughter Mary: pink, and dog Scruffy: green.

Is this a function? You bet it is! Even with no formula or equation showing!

We can put it in set circles or in a nice table like she did:

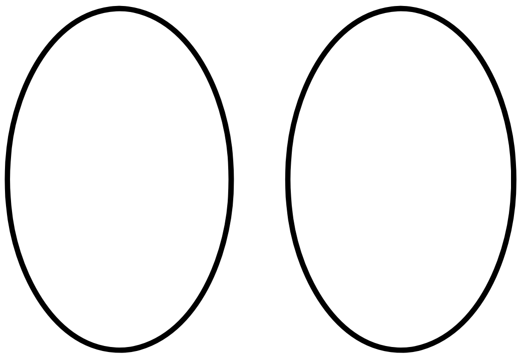
Family Member	Color
Mom	white
Dad	red
Son	blue
Daughter	pink
Dog	green

Our definition of a function is:

Let A and B be two sets. A function f from A to B is a rule or relation between A and B that assigns each element a in A to a unique element b in B.

Let's look at some set circles:

Here are two relations that are NOT functions



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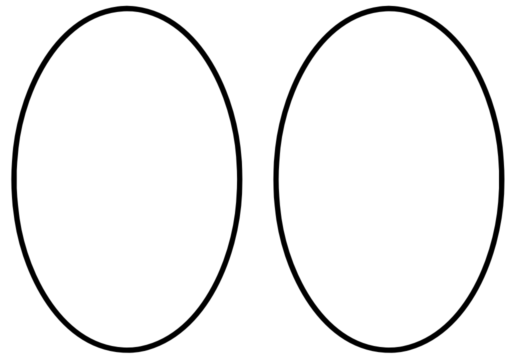
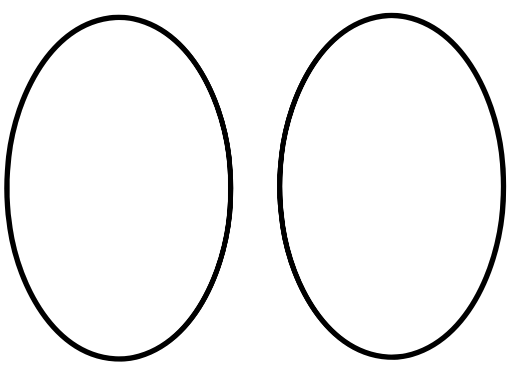


A definition of a function is

Let *A* and *B* be two sets. A function *f* from *A* to *B* is a rule or relation between A and B that assigns **each** element *a* in *A* to a **unique** element *b* in *B.*

The set *A* is usually called the **domain** of the function and *B* is the **range** of the function, a subset of the codomain.

Here are a pair of nice functions from the left set to the right set in each pair



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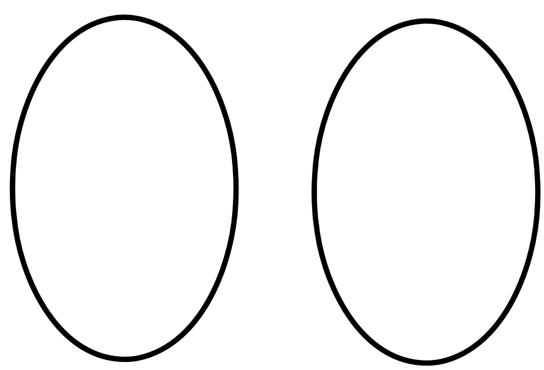


What is NOT said is just as important as what is said!
It's ok if two domain elements go to the same range element! Just not the reverse.

Popper 03 – Question 1

I. All functions are equations.

II. Each input has exactly one output.

III. All functions are relations.

IV. The input set is called the range.

V. All equations are functions.

Stepping **outside** our definition for just a moment:

There can be multiple sets contributing inputs (multi-variable functions) to get a single output.

In medicine:

Virginia Apgar devised the **Apgar score** in 1952 as a simple and replicable method to quickly and summarily assess the health of newborn children immediately after birth. Apgar was an anesthesiologist who developed the score in order to ascertain the effects of obstetric anesthesia on babies.

The Apgar scale is determined by evaluating the newborn baby on five simple criteria on a scale from zero to two, then summing up the five values thus obtained. The resulting Apgar score ranges from zero to 10. The five criteria are summarized using words chosen to form a backronym (**A**ppearance, **P**ulse, **G**rimace, **A**ctivity, **R**espiration). From each column in the table below, the infant is given a score of 0, 1 or 2. The scores are added up and the total sum is their Apgar score.

	Score of 0	**Score of 1**	**Score of 2**	**Component of Acronym**
Complexion	blue or pale all over	blue at extremities body pink	body and extremities pink	**A**ppearance
Pulse rate	absent	<100	>100	**P**ulse
Reflex	no response to stimulation	grimace/feeble cry when stimulated	cry or pull away when stimulated	**G**rimace
Activity	none	some flexion	flexed arms and legs that resist extension	**A**ctivity
Respiratory Effort	absent	weak, irregular, gasping	strong, lusty cry	**R**espiration

The test is generally done at one and five minutes after birth, and may be repeated later if the score is and remains low. Scores 7 and above are generally normal, 4 to 6 fairly low, and 3 and below are generally regarded as critically low.

*Source Wikipedia

In our class, we will work with single input functions and I really do want you to know that this is a proper subset of the set of all functions.

So where are we in the Big Picture:

single-valued functions $\subset$ functions $\subset$ relations

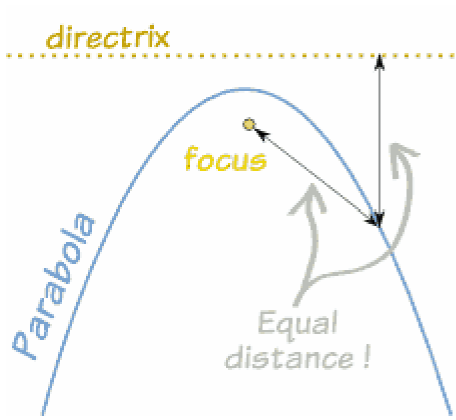
We will restrict ourselves further to relations and functions with inputs and outputs of real numbers…a proper subset of single-valued functions! Most of these can be graphed.

Let's look at some relations that are not functions and some relations that are functions.

We've already seen that circles are relations that are not functions. Now let's look at parabolas…not just college algebra parabolas, but geometric parabolas. Ones that open left, open right and open on a slant as well as up and down.

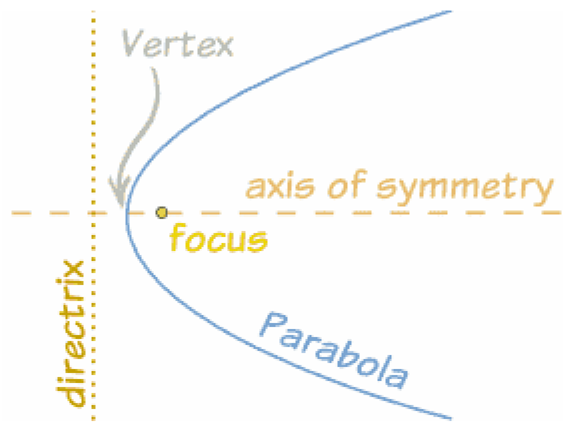
Parabolas:

Definition and Picture*



A parabola is a curve where any point is at an **equal distance** from:

- a fixed point (the **focus**), and
- a fixed straight line (the **directrix**)

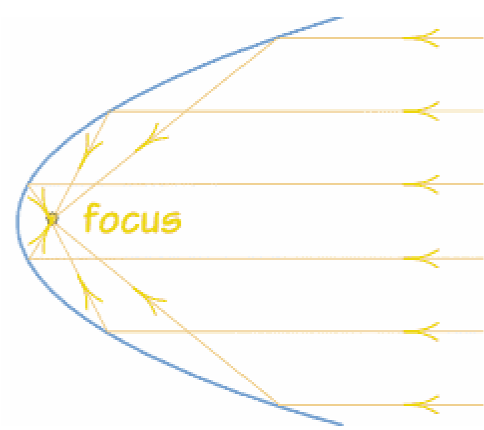


This one is NOT a function!

Vocabulary

Here are the important names:

- the **directrix** and **focus** (explained above)
- the **axis of symmetry** (goes through the focus, at right angles to the directrix)
- the **vertex** (where the parabola makes its sharpest turn) is halfway between the focus and directrix.



Reflector

And a parabola has this amazing property:

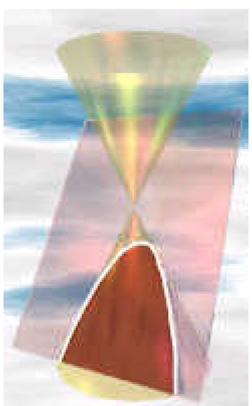
Any ray parallel to the axis of symmetry gets **reflected** off the surface straight **to the focus**.

And that explains why that dot is called the **focus** ...

... because that's where all the rays get focused!

So the parabola can be used for:

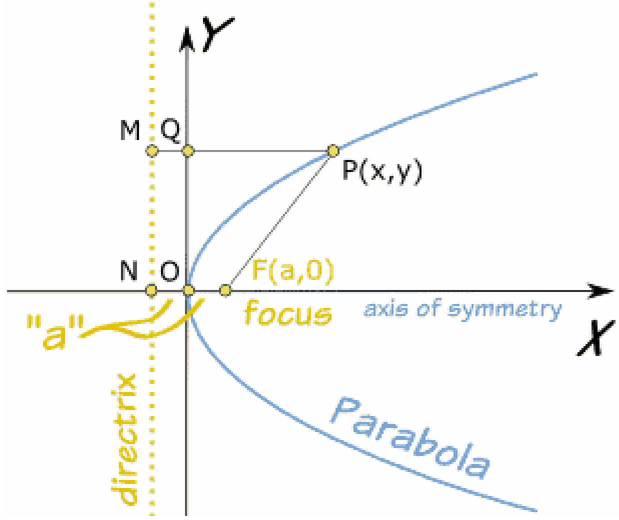
- satellite dishes,
- radar dishes,
- concentrating the sun's rays to make a hot spot,
- the reflector on spotlights and torches,



You can also get a parabola when you **slice through a cone** (the slice must be parallel to the side of the cone).

So the parabola is a conic section (a section of a cone).

Equations



If you place the parabola on the <u>cartesian coordinates</u> (x-y graph) with:

- its vertex at the origin **"O"** and
- its axis of symmetry lying on the x-axis,

then this curve is defined by:

$$y^2 = 4ax$$

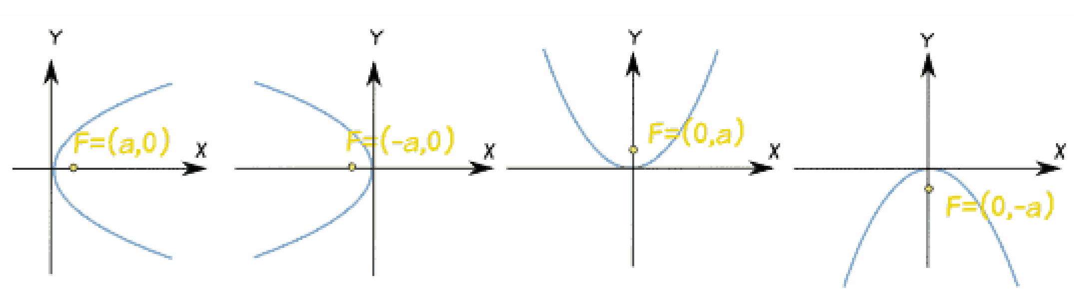
Example: Where is the focus in the equation y^2=5x ?

Converting $\mathbf{y^2 = 5x}$ to $\mathbf{y^2 = 4ax}$ form, we get $\mathbf{y^2 = 4\ (5/4)\ x}$,

so $\mathbf{a = 5/4}$, and the focus of y^2=5x is:

$$F = (a,0) = (5/4,0)$$

The equations of parabolas in different orientations are as follows:



$y^2 = 4ax$ | $y^2 = -4ax$ | $x^2 = 4ay$ | $x^2 = -4ay$

not a function | not a function | is a function | is a function

**http://www.mathsisfun.com/geometry/parabola.html GOOD SITE!
If the vertex is NOT at the origin, there will be an (h, k) for the vertex in the formula like this:

$$(y-k)^2 = 4a(x-h)$$

$$(x-h)^2 = 4a(y-k)$$

Now about those opening on a slant: there is an "xy" term in the equations of those and they look like:

So let's do another set diagram: relations and functions, and we'll put in parabolas

Working only with relations and functions that have graphs gives us a convenient test to see on paper if an equation is a function:

The Vertical Line Test (VLT):

If a vertical line through the graph touches at exactly one point, then the equation that produced the graph is a function. The vertical lines have the equation $x = k$ and the x's need to be those in the domain of the function. If you pick an x that is not in the domain, there won't be any graph point there to test with.

Now, DON'T be nice about this testing. Here are some typical errors:

A circle:

The s-graph:

The square root graph:

The domain and range of $f(x) = x^{\frac{1}{2}} = \sqrt{x}$

Some definitions of VLT accommodate this by saying "at most one intersection".

Popper 03 Question 2

Now, if we focus on functions with graphs, we can identify a proper subset of these special graphs that is a group!

First this new subset: one to one functions.

Then, the combining operation (composition).

Then we can check out the group structure.

One to one functions are those functions that pass the **Horizontal Line Test**.

Let's look at some of these and find their inverses:

[Note that these are a proper subset of graphable functions, so each has already passed the Vertical Line Test!]

Example 1

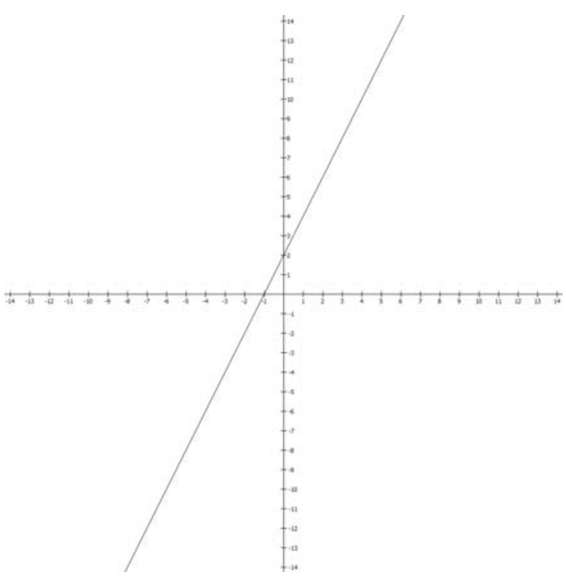
Lines with slopes that are not zero not infinity. Why these? Why the restriction? Can we be more efficient about it? How do you find that all important inverse function?

Let's do the HLT on a nice one…and then discuss what the HLT is telling us!

$f(x) = 2x + 2$

Intercepts Box

Graph: VLT, HLT



Domain and Range:

Inverse Function: SWAP X and Y

Popper 03, Question 3

Example 2
Discrete functions (i.e. sets of the right points)!

And their inverses.

{(1, 2), (2, 3), (3, 4), (4, 5)}

Domain

Range

SWAP X AND Y

Inverse function

Example 3 Rotated Hyperbolas

Traditional college algebra hyperbolas

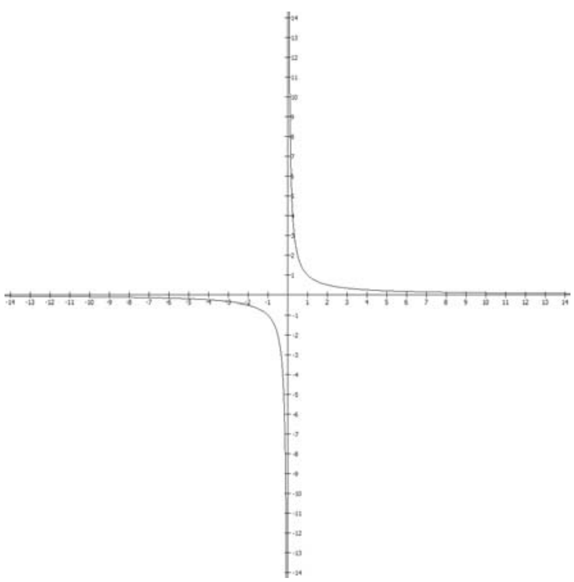
90° rotations $f(x) = \dfrac{1}{x}$ $xy = 1$

Domain

Range

Inverse function (SWAP X AND Y)

Graph



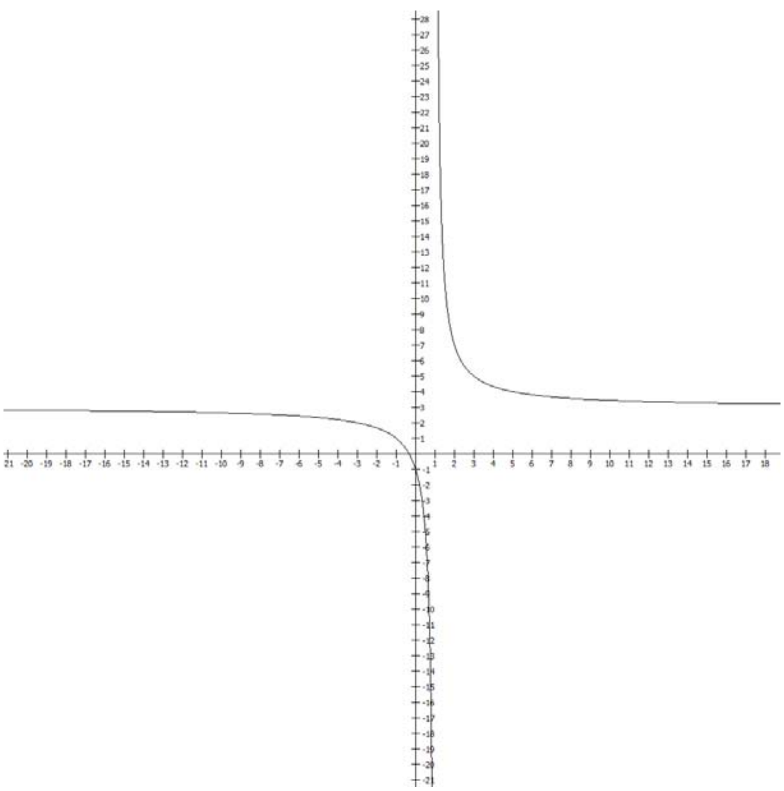
Second, more surprising rotated hyperbola

$f(x) = \dfrac{3x+1}{x-1}$ do the algebra to see the *xy* term.

Do the algebra to find the inverse function:

Domain and Range

Graph



Popper 03 Question 4

Example 4

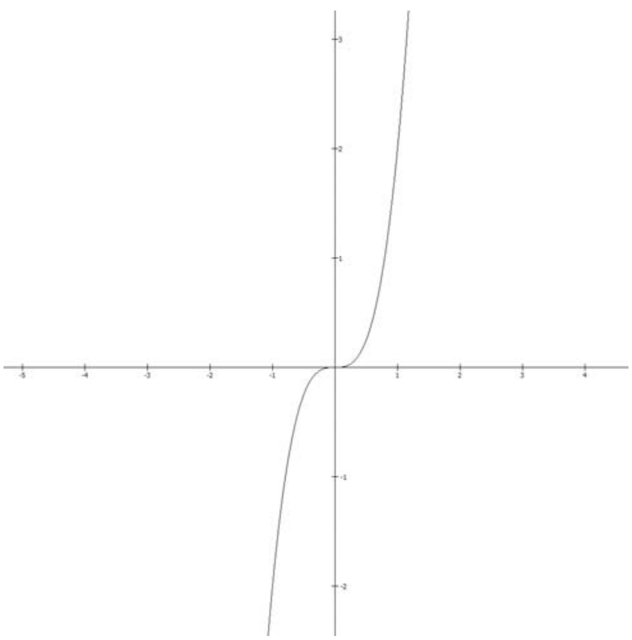
Power functions with odd exponents why NOT evens?

$f(x) = ax^{2n-1}$ n, a natural number

$f(x) = 2x$

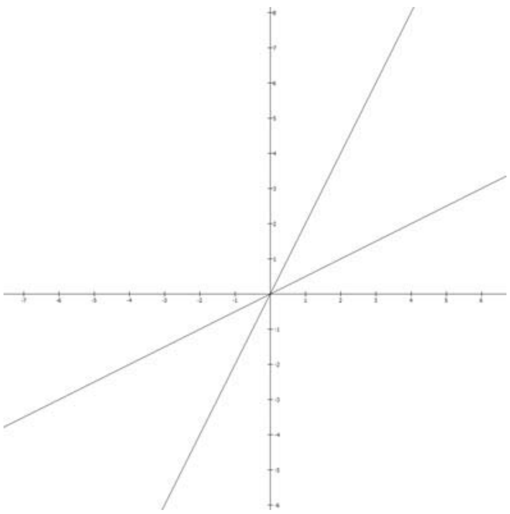
$f(x) = 2x^3$

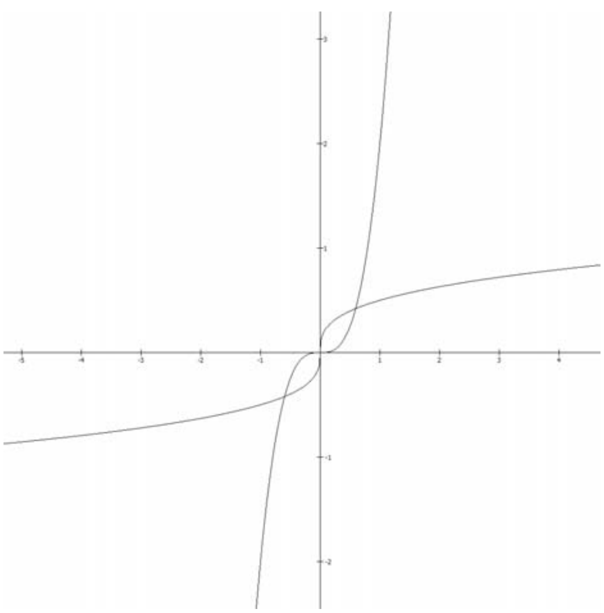
Cubic:



Inverses for both

Graphs of inverses:





Summary of inverse facts

One to one fuctions

(a, b) and (b, a)

Symmetry

Popper 03 Question 5

So we’ve got this proper subset of functions with graphs, one to one functions. We’ve got a nice reliable way to figure out which functions are in and which are out. What’s next? Combining functions!

Combining functions: add, subtract, multiply, divide, and composition. Composition is the one we’ll use in our group. We will run through combining fuctions with arithmetic first.

Let’s take:

$f(x) = x \qquad g(x) = 1$

Graph them and review what we know about them

Add them and subtract them and look at those graphs.

Take the ratio of 1 and x. What do we know about that graph?

Then we can look at $h(x) = 2$ $k(x) = x^2$ (note that $k(x)$ is just $f(x)$ times itself)

We can create a nice parabola with these. AND we can create another rotated hyperbola with this many functions. Let's look at these and review what we know.

$S(x) = x^2 + 2x + 1$ $T(x) = \dfrac{2x+1}{x-2}$

Popper 03 Question 6

$$f(x)=\frac{2x+1}{x-2}$$

Now let's look at composition. This is function-specific operation. You can't do it with numbers at all. When you compose two functions, you use one function as the input for the other. And the instructions on which is which are QUITE firm. Composition itself is signified by a little tiny hollow dot ("$\circ$") and it is read "after". So if I have two functions A and B and I write $A \circ B$, then I will do A after I do B to x. Now, $B \circ A$ is not the same instruction at all and most of the time, doing them in reverse order from the instructions will not result in the correct answer. Consider compose to be a "recipe"…when you're following a recipe, you need to do things in order and so it is with composing functions.

First we can practice composing with numbers as the input. Then we'll move into composing with functions as the input.

Example 1:

$f(x) = x^2 \qquad g(x) = 3 - x \qquad h(x) = \dfrac{1}{x} = x^{-1}$

If I take x = 2 and do $f \circ g(2)$ I will do *f* after I do *g* to 2. *g*(2) is 1. *f*(*g*(2) = f(1) = 1.

Note the input!

Now what about $g \circ f(2)$. I will do *g* after I do *f* to 2. *f*(2) = 4. *g*(4) = *g*(*f*(2)) = −1.

Look at the difference between the two answers! It really MATTERS how you read the instructions!

Now let's do $g \circ h \circ f(5)$

Do *g* after you do *h* after you do *f* to 5. Shorter but harder to read: $g(h(f(5)))$

The "hollow dot" instruction is easier to read. *f*(5) is 25…. *h*(25) = $\dfrac{1}{25}$ ….g(.04) = 2.96.

Popper 03 Question 7

$F(x) = x + 1$ $G(x) = x^2 + 2$

Can you see why saying $5 \circ 3$ makes no sense?

Composition is really only for situations where there is a way to input a number or a function into a PROCESS and get exactly one answer. This is why it's our combining operation of choice for our group.

Now, composition with functions as the inputs.

$f(x) = x^2 \qquad g(x) = 3 - x \qquad h(x) = \frac{1}{x} = x^{-1}$

$g \circ f(x)$ this instruction says do *f* to the whole domain of *f* and then do *g* to the resulting set!

Do *g* after you do *f* to x. Well, I look at f and note that the domain is all real numbers.

$f(x) = x^2 \qquad g(f(x)) = g(x^2) = 3 - x^2$

Let's check this numerically: if $x = 2$ did we get -1 above like we do here? Yes, we do.

What about $h \circ f \circ g(x)$?

$$g(x) = 3 - x$$
$$f(g(x)) = (3-x)^2 = 9 - 6x + x^2$$

$$h(f(g(x))) = \frac{1}{(3-x)^2} = \frac{1}{9 - 6x + x^2}$$

Oh, my, the care you have to take with this!

Popper 03 Question 8

The group is the set of one-to-one functions and composition for the combining operation.

The set is infinite in size, but is definitely a proper subset of the set of all functions.
Let's look at some examples of what is and is not in the set.

Lines with a slope that is not zero or infinity are in the set.
Let's look at why NOT zero or infinite slopes:

Power functions with odd powers of x are in the set. Why not even powers?

Roots like square root and cube root are in the group.

$\text{Tan}^{-1}(x)$ is in the group. (3305!) when you restrict the tangent function to one period.

What is the identity and how do you know it’s the identity?

Inverses

Associativity

Summary

What is the set?

What is the combining operation?

What is the identity function?

How do we know we have an inverse function for each element in our set?

How do we illustrate associativity? CAREFULLY!

Video 2 coming right up!